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Algebra of Deformed Differential Operators and Induced Integrable Toda Field Theory

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We build in this paper the algebra of q-deformed pseudo-differential operators, shown to be an essential step toward setting a q-deformed integrability program. In fact, using the results of this q-deformed algebra, we derive the q-analogues of the generalized KdV hierarchy. We focus in particular on the first leading orders of this q-deformed hierarchy, namely the q-KdV and q-Boussinesq integrable systems. We also present the q-generalization of the conformal transformations of the currents u_n, $n \geq 2$, and discuss the primary condition of the fields W_n, $n \geq 2$, by using the Volterra gauge group transformations for the q-covariant Lax operators. An induced su(n)-Toda(su(2)-Liouville) field theory construction is discussed and other important features are presented.

KEY WORDS: algebra; differential operators; Toda field theory.

1. INTRODUCTION

An interesting subject which has recently been studied from different point of views deals with the field of nonlinear integrable systems and their various higher and lower spin extensions (Bakas, 1989a,b; Bouwknegt and Schoutens, 1992; Das, 1987, Faddeev and Takhtajan, 1987; Fateev and Zamolodchikov, 1988; Jimbo and Miwa, 1990; Kupershmidt, 1990; Lax, 1968, 1975; Manin and Radul, 1985; Mathieu, 1988a,b; Saidi *et al.*, 1995a,b; Smit, 1990; Yamagishi, 1988; Zamolodchikov, 1985). These are exactly solvable models exhibiting a very rich structure in lower dimensions and are involved in many are as of mathematical

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physics. One recalls, for instance, the two-dimensional Toda (Liouville) fields theories (Alvarez-Gaumé and Gomez, 1991; Bakas, 1989a,b; Manin and Radul, 1985; Mansfield, 1982, 1983; Mathieu, 1988a,b; Olive and Turok, 1986; Smit, 1990; Yamagishi, 1988) and the KdV and KP hierarchy models (Bakas, 1989a,b; Das, 1989; Faddeev and Takhtajan, 1987; Jimbo and Miwa, 1990; Kupershmidt, 1990; Lax, 1968, 1975; Manin and Radul, 1985; Mathieu, 1988a,b; Smit, 1990; Yamagishi, 1988), both in the bosonic as well as in the supersymmetric case.

Nonlinear integrable models are associated to systems of nonlinear differential equations, which we can solve exactly. Mathamatically these models have become more fascinating by the introduction of some new concepts such as the infinite dimensional Lie (super) algebras (Cornwell, 1989; Humphreys, 1972; Kac, 1977), Kac-Moody algebras (Xian, 1991), W-algebras (Bouwknegt and Schoutens, 1992; Fateev and Zamolodchikov, 1988; Saidi *et al.*, 1995a,b; Zamolodchikov, 1985), quantum groups (Benkaddour *et al.*, 1998; Drinfeld, 1987; Faddeev, 1984; Jimbo, 1985, 1986; Wess and Zumino, 1990), and the theory of formal pseudo-differential operators (Bakas, 1989a,b; Das, 1989; Faddeev and Takhtajan, 1987; Jimbo and Miwa, 1990; Kupershmidt, 1990; Lax, 1968, 1975; Manin and Radul, 1985; Mathieu, 1988a,b; Saidi and Sedra, 1994a; Sedra, 1996; Smit, 1990; Yamagishi, 1988). Note, by the way, that techniques developed for the analysis of nonlinear integrable systems and quantum groups can be used to understand many features appearing in various problems of theoretical physics (Benkaddour and Saidi, 1999; Maroufi *et al.*, submitted; Saidi *et al.*, 1995a,b; Saidi and Sedra, 1993, 1994b,c; Sedra, 1998).

Recall that, since symmetries play an important role in physics, the principal task of quantum groups consists in extending these standard symmetries to the deformed ones, which might be used in physics as well.

Motivated by the relevance of both the generalized integrable KdV hierarchies and quantum deformations, we focus in this work to present a systematic study of bidimensional q-deformed nonlinear integrable models. We start then in Section 2 by presenting the algebra of q-deformed pseudo-differential operators. This provides the basic ingredients, which we need in the q-deformed integrability framework. Using these backgrounds, we will build, in Section 3, the q-analogues of the generalised KdV hierarchy. We will concentrate in particular on the first leading orders of this hierarchy, namely the q-KdV and q-Boussinesq integrable systems. In Section 4, we present the q-generalization of the conformal transformations of the currents $u_n, \geq 2$, and discuss the primary condition of the fields W_n, $n \geq 2$, by using the Volterra gauge group transformations for the q-convariant Lax operators. An induced su(n)-Toda(su(2)-Liouville) field theory construction is presented in Section 5. Other important results and some useful formulas are reported in Appendices A–E. Finally, we give conclusions.

2. THE ALGEBRA OF *q*-DEFORMED PSEUDO-DIFFERENTIAL OPERATORS

We start in this section from the well-known q-deformed derivation law, $\partial_z = 1 + qz\partial$ (Benkaddour *et al.*, 1998; Drinfeld, 1987; Faddeev, 1984; Jimbo, 1985, 1986; Wess and Zumino, 1990) and derive the q-analogue of the Leibnitz rule for both local and nonlocal differential operators. This result, which gives naturally the algebra of q-deformed (pseudo)-differential operators, will provide a way for generating a hierarchy of q-deformed Lax evolution equations.

2.1. The Ring of *q*-"Analytic" Currents

To start let us precise that the deformation parameter q we consider in this study is assumed to be a nonvanishing positive number.[4] Consider then the following q-deformed derivation rule (Benkaddour *et al.*, 1998; Drinfeld, 1987; Faddeev, 1984; Jimbo, 1985, 1986; Wess and Zumino, 1990):

$$\partial z = 1 + qz\partial, \tag{2.1}$$

where the symbol ∂ stands for the q-derivative $\partial_q \equiv \partial_q = \left(\frac{\partial}{\partial_z}\right)_q$.

As we already know, "conserved" currents are ingredients that we highly need in the programs of nonlinear integrable models and two-dimensional conformal field theory building. As we are interested in the present study to set up the basic tools toward extending such programs to q-analogue ones, we will try to describe first the ring of arbitrary q-"analytic" fields which we denote by R. Following the analysis developed in Saidi and Sedra (1994a) and Sedra (1996), this space describes a tensor algebra of fields of arbitrary conformal spin. This is a completely reducible infinite dimensional SO(2) Lorentz representation (module) that can be decomposed as

$$R = \underset{k\in z}{\oplus} R_k^{(0,0)}, \tag{2.2}$$

where $R_k^{(0,0)} = R_k$ are one-dimensional spin k-irreducible modules generated by the q-"analytic" fields $u_k(z)$ of "conformal" spin $k \in z$. The upper indices (0,0) carried by R, and that we shall drop whenever no confusion can arise, are special values of general indices (r, s) introduced in Saidi and Sedra (1994a) and Sedra (1996) and referring to the lowest and highest degrees of some pseudo-differential operators.

Inspring from the derivation law Eq. (2.1), we introduce in this ring a q-deformed derivative $\partial \equiv \partial q$ satisfying

$$\partial u_k(z) = u_k'(z) + \overline{q}^k u_k(z)\partial, \tag{2.3}$$

[4] This means that $q \in R^*$. However, if we suppose that $q \in C$, then we shall impose q to differ from the kth root of unity, i.e., $q^k \neq 1$, as we will show, for example, in Eqs (2.7) and (2.8). This requirement is justified by our need of consistency when we go to the standard limit $q = 1$.

with $\bar{q} = q^{-1}$ and $u_k' = \left(\frac{\partial u_k}{\partial z}\right)_q$ stands for the standard prime derivative. Note, by the way, the important fact that we have to distinguish between the prime derivative $u_k' = \partial u_k$ and the operator derivative $\partial u_k = (\partial u_k) + \bar{q}^k u_k \partial$. [Eq. (2.3)]. To illustrate what it means, consider the following examples:

Example 1: $u_{-k}(z) = z^k, k \geq 0$
For this choice of the field $u_{-k}(z)$, we drive the following expression:

$$(u_{-k})(z) = \left(\sum_{i=0}^{k-1} q^i\right) z^{k-1}, \tag{2.4}$$

as we can easily check by proceeding with the first leading terms $k = 0, 1, 2, \ldots$. Indeed, for $k = 0$, $(u_0)'\,(z) = 0$ and for $k = 1$ we have $u_{-1} \equiv z$, and by virtue of Eq. (2.1) we have

$$\begin{aligned} (u_{-1})'(z) \equiv (\partial u_{-1}) &= \partial u_{-1} - \bar{q}^{-1} u_{-1}\partial \\ &= \partial z - \bar{q}^{-1} z\partial \\ &= 1, \end{aligned} \tag{2.5}$$

which we can also derive from Eq. (2.4), with $\bar{q}^{-1} = q$. The nontrivial case is given by $k = 2$, such that $u_{-2} \equiv z^2$; therefore we have

$$\begin{aligned} (u_{-2})'(z) \equiv (\partial u_{-2}) &= \partial z^2 - \bar{q}^{-1} u_{-2}\partial \\ &= (1+q)z + q^2 z^2 \partial - \bar{q}^{-2} z^2 \partial \\ &= (1+q)z, \end{aligned} \tag{2.6}$$

which can also be easily seen from Eq. (2.4). These first leading cases clearly show from where the prime derivative formula (2.4) comes from.
The total Leibnitz derivative applied to $u_{-k}(z) = z^k$, $k \geq 0$, is simply derived using successive action of the deformed q-derivative $\partial \equiv \partial q$. We find

$$\partial z^k = \left(\sum_{i=0}^{k-1} q^i\right) z^{k-1} + q^k z^k \partial, \tag{2.7}$$

which justify, in some sense, the consistency of Eq. (2.4) in describing the “conformal spin” content of the analytic fields $u_k(z)$. Setting $k = 1$, one recovers in a natural way, the standard relation (2.1) just by setting $k = 1$. The second examples we consider is the following:

Example 2: $u_k(z) = z^{-k}, k \geq 1$
Corresponding relations are computed in the same way. We find

$$\partial z^{-k} = -\left(\sum_{i=1}^{k} \bar{q}^{-1}\right) z^{-k-1} + \bar{q}^{-k} z^{-k} \partial, \tag{2.8}$$

which reduces to

$$\partial z^{-1} = -\bar{q} z^{-2} + \bar{q} z^{-1} \partial \tag{2.9}$$

upon setting $k = 1$.

Now having introduced the ring R of analytic q-deformed currents, and shown how the q-deformed derivative acts, we are now in a position to introduce the space of q-deformed (pseudo)-differential operators.

2.2. The Space of q-deformed Lax Operators

Let $\Xi_m^{(r,s)}$ denote the space of q-deformed local differential operators, labeled by three quantum numbers m, r, and s defining respectively the conformal spin, the lowest and the highest degrees. Typical elements of this space are given by

$$L_m = \sum_{i=r}^{s} u_{m-i}(z)\partial^i, \quad r, s, m \in Z. \tag{2.10}$$

The symbol ∂ stands for the q-derivative and $u_{m-i}(z)$ are analytic fields of conformal spin$(m-i)$. The space $\Xi_m^{(r,s)}$ behaves then as a $(1+s-r)$-dimensional space generated by $L_m^{(r,s)} \equiv L_m$ and whose space decomposition is given by the linear sum

$$\Xi_m^{(r,s)} = \bigoplus_{i=r}^{s} \Xi_m^{(i,i)}, \tag{2.11}$$

with

$$\Xi_m^{i,i} = R_m \otimes \partial^i. \tag{2.12}$$

A special example of the space $\Xi_m^{(r,s)}$ is given by $R_m \equiv \Xi_m^{(0,0)}$ [Eq. (2.2)], the set of analytic fields $u_m(z)$ introduced previously and $\partial^i \equiv \partial_q^i$ is the ith q-derivative. A natural example of Eq. (2.10) is given by the q-deformed Hill operator

$$L_2 = \partial^2 + u_2(z), \tag{2.13}$$

which will play an important role in the study of the q-deformed KdV equation and the associated "conformal" q-Liouville field theory.

A result concerning the algebra $\Xi_m^{(r,s)}$ is the derivation of the q-Leibnitz rule for local q-differential operators. Focusing to derive the general formula, let us start first by examining the first leading orders. Iteration processing applied to Eq. (2.3) gives the following relations:

$$\begin{aligned}
\partial u_k(z) &= u_k'(z) + \bar{q}^k u_k(z)\partial \\
\partial^2 u_k(z) &= u_k''(z) + \bar{q}^k(1+\bar{q})u_k'(z)\partial + \bar{q}^{2k} u_k(z)\partial^2 \\
\partial^3 u_k(z) &= u_k'''(z) + \bar{q}^k(1+\bar{q}+\bar{q}^2)u_k''(z)\partial + \bar{q}^{2k}(1+\bar{q}+\bar{q}^2)u_k^i(z)\partial^2
\end{aligned} \tag{2.14}$$

$$+\bar{q}^{3k}u_k(z)\partial^3$$
$$\vdots$$

The crucial point was the observation that[5] these higher first-order derivations formulas can be summarized into the following general Leibnitz rule:

$$\partial^{v}u_k(z) = \sum_{j=0}^{p} \bar{q}^{(p-j)k}\chi_p^j(q)u_k^{(j)}(z)\partial^{p-j}, \quad p \geq 0, \tag{2.15a}$$

where $\chi_p^j(q)$ are q-coefficient functions that we have introduced such that

$$\chi_p^0(q) = \chi_p^p(q) = 1 \tag{2.15b}$$

and

$$\begin{aligned}\chi_p^j(q) = 1 &+ \bar{q}^j \sum_{m_1=0}^{j-1} q^{m_1} + \bar{q}^{2j} \sum_{m_1=0}^{j-1}\sum_{m_2=0}^{j-1-m_1} q^{2m_1+m_2} \\ &+\bar{q}^{3j}\sum_{m_1=0}^{j-1}\sum_{m_2=0}^{(j-1-m_1)}\sum_{m_3=0}^{(j-1-(m_1+m_2))} q^{3m_1+2m_2+m_3} \\ &+\cdots \\ &+\bar{q}^{(p-j)j}\sum_{m_1=0}^{j-1}\sum_{m_2=0}^{j-1-m_1}\cdots\sum_{m_{p-j}=0}^{j-1-\sum_{i=1}^{p-j-1}m_i} q^{\sum_{\beta=0}^{p-j-1}(p-j-1-\beta)m_{\beta+1}}\end{aligned} \tag{2.15c}$$

for $1 \leq j \leq p-1$. Some remarks are in order:

1. From the q-Leibnitz rule (2.15a), one can deduce the q-analogue of the standard binomial coefficients C_p^j as follows:

$$\begin{aligned} C_p^0 &\xrightarrow{q} \bar{q}^{pk}\chi_p^0(q) \equiv \bar{q}^{pk} \\ C_p^p &\xrightarrow{q} \chi_p^p(q) = 1 \end{aligned} \tag{2.16a}$$

   and for $1 \geq j \geq p-1$

$$C_p^j \xrightarrow{q} \bar{q}^{(p-j)k}\chi_p^j(q) \tag{2.16b}$$

2. Setting $q = 1$, the local Leibnitz rule (2.15a) reduces naturally to the standard derivation law

$$\partial^p u_k(z) = \sum_{j=0}^{p} C_p^j u_k^{(j)}(z)\partial^{p-j}, \quad p \geq 0, \tag{2.17a}$$

[5] This observation was possible after performing several nontrivial algebraic manipulations toward writing Eqs. (2.14) in a compact form.

giving rise to the following useful relations

$$\begin{aligned}\chi_p^0(1) &= C_p^0 = 1\\ \chi_p^p(1) &= C_p^p = 1\end{aligned} \tag{2.17b}$$

and for $1 \leq j \leq p-1$

$$\begin{aligned} C_p^j = \chi_p^j(1) = 1 + j + \frac{j(j+1)}{2}\\ + \sum_{m_1=0}^{j-1} \sum_{m_2=0}^{(j-1-m_1)} \sum_{m_3=0}^{(j-1-m_1-m_2)} 1\\ + \cdots + \sum_{m_1=0}^{j-1} \sum_{m_2=0}^{j-1-m_1} \cdots \sum_{m_{p-j=0}}^{j-1-\sum_{j=1}^{p-j-1}} 1 \end{aligned} \tag{2.17c}$$

3. As we can easily check, Eq. (2.15c) is a sum of $(p-j+1)$ objects starting from the value 1 which corresponds to set $(j=p)$ with zero summation. In each term of the remaining $(p-j)$ objects, we have a product of (n) summation $\sum_{m_1=0} \sum_{m_2=0} \cdots \sum_{m_n=0}$ with $1 \leq n \leq p-j$. This structure is useful in the standard limit $q=1$, recovering then the explicit form [Eq. (2.17c)] of the well-known binomial coefficient $C_p^j = \frac{P!}{(P-j)!j!}$.

Moreover, Eq. (2.10) which is well defined for local differential operators with $s \geq r \geq 0$, may be extended by the negative integers (nonlocal ones) by introducing q-deformed pseudo-differential operators of the type ∂_q^{-q}, $p \geq 1$, whose action on the fields $u_k(z)$ of conformal spin $k \in z$ is constrained to satisfy

$$\partial^p \partial^{-p} u_k(z) = \partial^{-p} \partial^p u_k(z) = u_k(z). \tag{2.18}$$

Following the same analysis developed previously, we derive the following formulas:

$$\begin{aligned}
\partial^{-1} u_k(z) &= \sum_{i=0}^{\infty} (-)^i q^{\left(k(i+1)+\frac{i(i+1)}{2}\right)} u_k^{(i)}(z) \partial^{-i-1}\\
\partial^{-2} u_k(z) &= \sum_{i=0}^{\infty} (-)^i q^{\left[k(i+2)+\frac{i(i+1)}{2}\right]} \left(\sum_{j=0}^{i} q^j \right) u_k^{(i)}(z) \partial^{-2-i},\\
\partial^{-3} u_k(z) &= \sum_{i=0}^{\infty} (-)^i q^{\left[k(i+3)+\frac{i(i+1)}{2}\right]} \left(\sum_{j_1=0}^{i} \sum_{j_2=0}^{j_1} q^{j_1+j_2} \right) u_k^{(i)} \partial^{-3-i},\\
&\vdots
\end{aligned} \tag{2.19}$$

From these first leading formulas, we extract the following nonlocal Leibnitz rule:

$$\partial^{-p}u_k(z) = \sum_{i=0}^{\infty}(-)^i q^{\left[k(i+p)+\frac{i(i+1)}{2}\right]}\left[\sum_{j_1=0}^{i}\sum_{j_2=0}^{ji}\cdots\sum_{j_{p-1}=0}^{j_{p-2}} q^{\sum_{m=1}^{p-1} j_m}\right] u_k^{(i)}(z)\partial^{-p-i} \tag{2.20}$$

Here we also remark that, for a fixed value of $p \geq 1$, we have a q-deformed binomial coefficient given by a product of $(p-1)$ summation $\sum_{m_1=0}\cdots\sum_{m_{p-1}=0}$. Setting $q=1$, one recovers the standard Leibnitz rule for nonlocal differential operators, namely

$$\partial^{-p}u_k(z) = \sum_{i=0}^{p}(-)^i C_{i+p-1}^{i} u_k^{(i)}\partial^{-p-1} \tag{2.21}$$

for $p \geq 1$, with the identity relation

$$C_{i+p-1}^{i} = \sum_{J_1=0}^{i}\sum_{j_2=0}^{ji}\cdots\sum_{j_{p-1=0}}^{j_{p-2}} 1 \tag{2.22}$$

coinciding exactly with $\chi_{i+p-1}^{i}(1)$ as we can easily learn from Eq. (2.17b). Other important results are reported in Appendix A.

Up to now, we have introduced the ring R of analytic functions and constructed the space of arbitrary q-deformed Lax operators by deriving the generalized q-Leibnitz rules. The next task is to see how we can apply the obtained results to study some important features of nonlinear integrable systems and conformal symmetry. Special examples, namely the Liouville field theory and the KdV equation as well as their extensions, will be considered.

3. GENERALIZED *q*-DEFORMED KdV HIERARCHY

In this section we propose to apply the results found previously to build the q-analogues of the KdV-hierarchy systems. We will consider in particular the first leading orders of this hierarchy, namely the KdV and Boussinesq integrable systems.

Let us consider the q-deformed KdV Lax operator

$$L_2 = \partial^2 + U_2, \tag{3.1}$$

which belongs to the coset space $\frac{\Xi_2^{(0,2)}}{\Xi_2^{(1,1)}}$, for which we have $u_0 = 1$ and $u_1 = 0$. As known from standard references in nonlinear integrable models (Bakas, 1989a,b; Das, 1989; Faddeev and Takhtajan, 1987, Jimbo and Miwa, 1990; Kupershmidt 1990; Lax 1968, 1975; Manin and Radul, 1985; Mathieu, 1988a,b; Smit, 1990;

Yamagishi, 1988), we can set by analogy

$$\frac{\partial L_2}{\partial t_{2n+1}} = (H_{2n+1}, L_2)_q, \tag{3.2}$$

which gives the nth evolution equation of the q-deformed KdV-hierarchy with

$$H_{2n+1} = \left(L_2^{\frac{2n+1}{2}}\right)_+. \tag{3.3}$$

The index "+" in Eq. (3.3) stands for the local part of the deformed pseudo-differential operator $L_2^{\frac{2n+1}{2}}$ defined as

$$L_2^{\frac{2n+1}{2}} = L_2^{\frac{1}{2}} L_2^n. \tag{3.4}$$

$L_2^{1/2}$ is just the half power of the q-KdV Lax operator introduced in Eq. (3.1). It describes a q-deformed pseudo-differential operator of dimension $2 \times \frac{1}{2} = 1$. The nonlinear q-deformed pseudo-differential operator $L_2^{2n+1/2}$ is just the $(2n+1)$th power of $L_2^{1/2}$. The standard method used to construct such kinds of operators can be found in one of the references cited in [1]. To work out explicitly H_{2n+1} we first need to compute $L_2^{1/2}$. Using dimensional arguments we assume that $L_2^{1/2}$ takes the following form:

$$L_2^{\frac{1}{2}} = \partial + a(q)u_2\partial^{-1} + b(q)u_2'\partial^{-2} + (c(q)u_2'' - d(q)u_2^2)\partial^{-3} + \cdots, \tag{3.5}$$

where the first leading coefficients a, b, c, and d are required to satisfy

$$L_2 = L_2^{\frac{1}{2}} L_2^{\frac{1}{2}}. \tag{3.6}$$

Using this requirement, we find explicitly

$$\begin{aligned} a(q) &= \frac{1}{1+\overline{q}^2} \\ b(q) &= \frac{1}{(1+\overline{q}^3)(1+\overline{q}^2)} \\ c(q) &= \frac{1}{(1+\overline{q}^2)(1+\overline{q}^3)(1+\overline{q}^4)} \\ d(q) &= \frac{q^2}{(1+\overline{q}^2)^2(1+\overline{q}^4)}. \end{aligned} \tag{3.7}$$

Later on, we will introduce the dot on the analytic fields $\dot{u}_2$ to describe the derivation with respect to time coordinates while the prime derivative is already introduced in Eq. (2.3) to denote the derivation with respect to the space variable z.

Furthermore, the bracket introduced in Eq. (3.2) is nothing but the q-deformed commutator, which we define as

$$[f\partial^n, g\partial^m]_q = \overline{q}^{m\tilde{f}} f\partial^n g\partial^m - \overline{q}^{n\tilde{g}} g\partial^m f\partial^n, \tag{3.8}$$

where f and g are two arbitrary functions of conformal spin $\tilde{f}$ and $\tilde{g}$. Setting $n = 0$, Eq. (3.2) becomes

$$\frac{\partial L_2}{\partial t_1} = [H_1, L_2]_q, \tag{3.9}$$

where $H_1 = (L_2^{\frac{1}{2}})_+ = \partial$. We also show that Eq. (3.9) corresponds simply to the chiral wave equation

$$\dot{u}_2 = u_2', \tag{3.10}$$

which means the equality of dimensions $[t_1] = [z]$. For $n = 1$, one has

$$\frac{\partial L_2}{\partial t_3} = \left[\left(L_2^{\frac{3}{2}}\right)_+, L_2\right]_q, \tag{3.11}$$

where $(L_2^{\frac{3}{2}})_+$, explicitly given by

$$\left(L_2^{\frac{3}{2}}\right)_+ \partial^3 + (\overline{q}^2 + a(q))u_2\partial + (1 + b(q))u_2', \tag{3.12}$$

is the q-deformed Hamiltonian operator associated with the q-Virasoro algebra.

Injecting this expression into Eq. (3.11) we can extract a nonlinear differential equation giving the evolution of the q-spin-2 current u_2, once some easy algebraic manipulations are done. Indeed, identifying the r.h.s. and l.h.s. terms of Eq. (3.11), we shall impose some terms of the r.h.s to vanish. We then obtain the following differential equation:

$$u_2 = A(q)u_2u_2' + B(q)u_2''', \tag{3.13}$$

where $A(q)$ and $B(q)$ are two constrained, q-dependent coefficients functions, which can be determined. Simple computations then lead to

$$\begin{aligned} A(q) &= \frac{1 + \overline{q} + \overline{q}^4}{1 + q^2} \\ B(q) &= -\frac{1 + \overline{q} + \overline{q}^2}{(\overline{q} + 1)^2}. \end{aligned} \tag{3.14}$$

This nonlinear differential equation is nothing but the q-deformed KdV system

$$\dot{u}_2 = \left(\frac{1 + \overline{q} + \overline{q}^4}{1 + \overline{q}^2}\right) u_2u_2' - \frac{1 + \overline{q} + \overline{q}^2}{(1 + \overline{q})^2}u_2''', \tag{3.15}$$

which coincides in the classical limit with the well-known KdV integrable system (Das, 1989)

$$u_2 = \frac{3}{2}u_2 u_2' - \frac{3}{4}u_2''', \tag{3.16}$$

and associated to the Hamiltonian differential operator

$$(L_2^{\frac{3}{2}})_+ = \partial^3 + \frac{3}{2}u_2\partial + \frac{3}{4}u_2'. \tag{3.17}$$

The same computations hold for the q-deformed Boussinesq equation. For more details concerning the results obtained for this system, we refer to Appendix B. Note finally that the deformed KdV hierarchy discussed in this paper is based on the structure of the algebra of q-pseudo-differential operators as described previously. Other q-deformation of this hierarchy are also possible; as an example we refer the reader to Frankel (1996).

4. CONFORMAL TRANSFORMATIONS AND q–W CURRENTS

We start this section by presenting the conformal transformation of the spin-2 current $u_2(Z)$ of the q-KdV hierarchy and give later the general relations for the higher spin conformal currents $u_n(z)$, $n \geq 2$. We also discuss the primary condition of the fields W_n, $n \geq 2$, by using the Volterra gauge group transformations for the q-covariant Lax operators.

4.1. q-Generalized Conformal Transformations

Let

$$L_2 = \partial^2 + u_2 \tag{4.1}$$

be the Lax operator of the q-KdV hierarchy discussed in Section 3. Now we want to show how the spin-2 conformal current $u_2(z)$ transforms under a conformal transformation,

$$z \to \tilde{z} = f(z). \tag{4.2}$$

Under such a transformation, we assume that the q-deformed KdV Lax operator (4.1) transforms as (Bakas, 1989a,b; Di-Francesco *et al.*, 1991; Manin and Radul, 1985; Mathieu, 1988a,b; Smit, 1990; Yamagishi, 1988)

$$L_2(u(z)) \to \tilde{L}_2(\tilde{u}_2(z)) = \psi^{-\frac{3}{2}} L_2(u_2(z))\psi^{\frac{1}{2}}, \tag{4.3}$$

where $\psi = \frac{\partial z}{\partial \tilde{z}}$. The choice of Ψ-powers in Eq. (4.3) is dictated by the fact that $L_2(u(z))$ maps densities of degree $(-\frac{1}{2})$ to densities of degree $\left(+\frac{3}{2}\right)$. We have

$$\partial \to \tilde{\partial} = \psi\partial, \tag{4.4}$$

which imply

$$\tilde{\partial}^2 = \psi\psi'\partial + \psi^2\partial^2. \tag{4.5}$$

Using straightforward computations, we find

$$\psi^{\frac{3}{2}} L_2 \psi^{\frac{1}{2}} = \psi^2\partial^2 + \frac{1}{2}(1+\overline{q})\psi'\psi\partial + \psi^2 u_2 + \frac{1}{2}\left(\psi''\psi - \frac{1}{2}\overline{q}(\psi')^2\right), \tag{4.6}$$

from which we can easily derive the following result:

$$\tilde{L}_2(\tilde{u}(\tilde{z})) = \tilde{\partial}^2 + \frac{\overline{q}-1}{2}\psi'\tilde{\partial} + \tilde{u}_2. \tag{4.7}$$

This clearly shows how the conformal transformation violates the standard convariantization property in the case of q-Lax operators. However, for $q = 1$, we recover this property naturally, since the coefficient term of $\tilde{\partial}$ in Eq. (4.7) vanishes as is proportional to $\frac{\overline{q}-1}{2}$.

Using the identification Eq. (4.7), we obtain the following conformal transformation for the field $u_2(z)$:

$$u_2(z) = \psi^{-2}\tilde{u}_2(\tilde{z}) - \frac{1}{2} S_u^{(2)}(q, \psi), \tag{4.8}$$

where we denote by $S_{u_2}^{(2)}(q, \psi)$ the q-Shwarzian derivative associated with the current u_2 and defined as

$$S_{u_2}^{(2)}(q, \psi) = \frac{\psi''}{\psi} - \frac{1}{2}\overline{q}\left(\frac{\psi'}{\psi}\right)^2. \tag{4.9}$$

The upper index "(2)" in $S_{u_2}^{(2)}$ stands for the order of the q-KdV hierarchy.

Furthermore, Eq. (4.8) shows that $u_2(z)$ transforms, as a field of conformal spin 2, up to an anomalous term $Su_2{}^{(2)}(q, \psi)$ exactly like the energy–momentum tensor of two-dimensional conformal fields theories.

The second example we consider is the q-Boussinesq hierarchy associated with the q-deformed Lax operator

$$L_3(u_2, u_3) = \partial^3 + u_2\partial + u_3. \tag{4.10}$$

Similarly, the conformal transformation (4.2) implies in this case

$$L_3(u_2, u_3) \to \tilde{L}_3(\tilde{u}_2, \tilde{u}_3) = \psi^2 L_3(u_2, u_3)\psi, \tag{4.11}$$

leading to the following result:

$$\begin{aligned} \psi^2 L_3 \psi = \psi^3\partial^3 + (1+\overline{q}+\overline{q}^2)\psi^2\psi'\partial^2 + \{(1+\overline{q}+\overline{q}^2)\psi^2\psi'' \\ + u_2\psi^3\}\partial + u_3\psi^3 + u_2\psi^2\psi'\psi^2\psi''', \end{aligned} \tag{4.12}$$

with

$$\tilde{L}_3(\tilde{u}, \tilde{u}_3) = \tilde{\partial}^3 + (\overline{q}^2 - 1)\psi'\tilde{\partial}^2 + \tilde{u}_2\tilde{\partial} + \tilde{u}_3. \tag{4.13}$$

Using again the identification (4.11), we obtain the following results:

$$u_2 = \psi^{-2}\tilde{u}_2 + S^{(3)}_{u_2}(q, \psi)(a)$$
$$u_3 = \psi^{-3}\tilde{u}_3 - \frac{\psi'}{\psi}\tilde{u}_2 + S^{(3)}_{u_2}(q, \psi)(b), \tag{4.14}$$

where $S^{(3)}_{u_2}$ and $S^{(3)}_{u_3}$ are the q-Shwarzian derivatives associated respectively with the conformal u_2 and u_3. They are given by

$$\begin{aligned} S^{(3)}_{u_2}(q, \psi) &= \bar{q}^2\left(\frac{\psi'}{\psi}\right)^2 - \bar{q}(\bar{q}+1)\frac{\psi''}{\psi} \\ S^{(3)}_{u_3}(q, \psi) &= \frac{\psi'''}{\psi} + \frac{\psi'}{\psi}S^{(3)}_{u_2}(q, \psi). \end{aligned} \tag{4.15}$$

Note by the way that $S^{(3)}_{u_2}$ and $S^{(3)}_{u_3}$ are shown to relate as follows:

$$\partial S^{(3)}_{u_2} + \bar{q}(\bar{q}+1)S^{(3)}_{u_3} = 0. \tag{4.16}$$

As suspected for $q = 1$, one can find the standard formulas given by (Bakas, 1989a,b; Di-Francesco *et al.*, 1991; Manin and Radul, 1985; Mathieu, 1988a,b; Smit, 1990; Yamagishi, 1988)

$$\begin{aligned} u_2 &= \psi^{-2}\tilde{u}_2 - S^{(3)}_{u_2}(1, \psi) \\ u_3 &= \psi^{-3}\tilde{u}_3 - \frac{\psi'}{\psi_3}\tilde{u}_2 - S^{(3)}_{u_3}(1, \psi), \end{aligned} \tag{4.17}$$

with

$$\begin{aligned} S^{(3)}_{u_2}(1, \psi) &= \left(\frac{\psi'}{\psi}\right)^2 - 2\frac{\psi''}{\psi} \\ S^{(3)}_{u_3}(1, \psi) &= \frac{\psi'''}{\psi} + \frac{\psi'}{\psi}S^{(3)}_{u_2}(1, \psi), \end{aligned} \tag{4.18}$$

and

$$\partial S^{(3)}_{u_2} + 2S^{(3)}_{u_3} = 0. \tag{4.19}$$

The presence of the anomalous term in Eq. (4.14b) can be removed by a convenient basis choice, namely the primary basis, which we will discuss later on.

Having given explicitly the conformal transformation of the currents u_2 and u_3 of conformal spin 2 and 3 respectively, we now focus to generalize these results to higher conformal spin currents $u_n(z)$, $n = 2, 3 \ldots$.

Let

$$L_n[u] = \partial^n + \sum_{i=0}^{n-2} u_{n-1}\partial^i \tag{4.20}$$

be the higher order Lax operator involving $(n-1)$ conformal currents with $u_0 = 1$ and $u_1 = 0$ and where $\partial = \partial_q$. Under the conformal transformation (4.2), this Lax operator is assumed to transform as

$$L_n[u] \to \tilde{L}_n[\tilde{u}] = \psi^{\frac{n+1}{2}} L_n[u] \psi^{\frac{n-1}{2}}. \tag{4.21}$$

Similar to the previous study, the structure of the Lax operator $L_n[u]$ [Eq. (4.20)] is broken under the conformal transformation. We find in general

$$\tilde{L}_n[\tilde{u}] = \tilde{\partial}^n + A\psi' \tilde{\partial}^{n-1} + \sum_{i=0}^{n-2} \tilde{u}_{n-1} \tilde{\partial}^i, \tag{4.22}$$

where A is an arbitrary Lorentz scalar function which we will precise.

To determine $\tilde{L}_n$, we need to compute explicitly $\tilde{\partial^n}$. Starting from Eq. (4.4) and using simply algebraic manipulation, we find the following results:

$$\tilde{\partial}^n = \sum_{i=1}^{n} M_i^n \partial^i, \tag{4.23}$$

where M_i^n are functions of conformal spin $(n-i)$, which we can summarize as follows:

$$\begin{aligned} M_n^n &= \psi^n \\ M_1^n &= \psi \partial M_1^{n-1} \\ M_i^n &= \psi[M_{i-1}^{n-1} \overline{q}^{(n-i)} + \partial M_i^{n-1}], \quad 2 \le i \le n-1. \end{aligned} \tag{4.24}$$

Substituting these relations into Eq. (4.22) we find

$$\tilde{L}_n = \sum_{i=0}^{n} X_i(A, M, \psi) \partial^{n-1}, \tag{4.25}$$

where

$$X_i(A, M, \psi) = \sum_{j=0}^{i} \tilde{u}_j M_{n-i}^{n-j}. \tag{4.26}$$

On the other hand, simply performing algebraic calculations show that the r.h.s. of Eq. (4.21) lead to

$$\psi^{\frac{n+1}{2}} L_n[u] \psi^{\frac{n-1}{2}} = \psi^{\frac{n+1}{2}} \sum_{i=0}^{n} \left(\sum_{j=0}^{i} u_{i-j} X_{n-j+i}^{j}(q) \left(\psi^{\frac{n-1}{2}} \right) \right) \partial^{n-i}. \tag{4.27}$$

Identifying then Eq. (4.25) with Eq. (4.27) we find

$$A(q, \psi) = \frac{\psi^{1-n}}{\psi'} \left[\chi_n^1(q) \psi^{\frac{n+1}{2}} \left(\psi^{\frac{n-1}{2}} \right)' - M_{n-1}^n \right], \tag{4.28}$$

with

$$\chi_n^1(q) = \sum_{i=0}^{n-1} \bar{q}^i$$
$$\chi_n^0 = \chi_n^n = 1, \tag{4.29}$$

and

$$u_i = \psi^{-n} \left\{ M_{n-i}^n + \sum_{j=1}^{i} \left[\tilde{u}_j M_{n-i}^{n-j} - \psi^{\frac{n+1}{2}} u_{i-j} \chi_{n-i+j}^j(q) \left(\psi^{\frac{n+1}{2}} \right)^{(j)} \right] \right\}$$
$$\text{for } 0 \leq i \leq n. \tag{4.30}$$

We then clearly show how to transform the conformal currents u_i, $i \geq 2$, under Eq. (4.2). The first thing we learn from these results is the dependence on the q-parameter, which once coincides with $q = 1$ leads to the standard formulas. To illustrate the obtained results, we consider two particular examples discussed previously, namely the q-KdV and q-Boussinesq integrable models described respectively by $L_2(u)$ and $L_3(u)$.

The former is easily obtained by setting $n = 2$ into Eqs. (4.28)–(4.30), which recover Eqs. (4.8) and (4.9) exactly with

$$A = \frac{\bar{q} - 1}{2} \tag{4.31}$$

Similarly, Eqs. (4.14) and (4.15) are obtained by setting $n = 3$ Eqs. (4.27) and (4.28) with

$$A = \bar{q}^2 - 1. \tag{4.32}$$

4.2. Volterra Gauge Group Transformation and q–W Currents

In the framework to generalise the conformal transformations to the q-deformed case, we found, in addition to new features, the presence of anomalous terms at the level of the conformal current $u_3, u_4, \ldots, u_n$ [Eq. (3.30)].

Our idea is to consider a Volterra gauge group transformation associated to an "orbit" in which no such anomalous terms can appear. We start first by recalling the Volterra gauge group symmetry. This is a symmetry group whose typical elements are given by the Lorentz scalar q-pseudo-differential operators (Rachidi, xxxx)

$$K[a] = 1 + \sum_{i \geq 1} a_i(z) \partial^{-1}, \tag{4.33}$$

where $a_i(z)$ are arbitrary analytic functions of conformal spin $i = 1, 2, 3, \ldots$. These functions, to which we shall refer hereafter to as the Volterra gauge parameters, can be expressed in terms of the residue operation as

$$a_i(z) = \mathrm{Res}^{(K(a)\partial^{i-1})}, \tag{4.34a}$$

where

$$\mathrm{Res}^{\partial^i} = \delta_{i+1,0}, \tag{4.34b}$$

and for a given function $f(z)$, we recall that we have Eq. (2.3):

$$\partial f(z) = f(z) + \bar{q}^{\tilde{f}} f(z)\partial. \tag{4.35}$$

Next, we will apply this Volterra gauge group symmetry to the algebra of q-Lax operators (4.20) via the following relation:

$$L_n(u) \to L_n(w) = K^{-1}(a)L_n(u)K(a), \tag{4.36}$$

where $L_n(w)$ is the transform of $L_n(u)$ under the Volterra group action with $w = w(a, u)$ is a function which depends on the Volterra parameter a_i and the u-fields. Moreover, Eq. (4.36) shows that the u-currents may be expressed completely in terms of the Volterra gauge parameters a_i and their kth derivatives. However, solving Eq. (4.36), one finds that the new fields w_i are polynomials in the old u-fields and the Volterra parameters and their derivatives.

Making the appropriate choices of the Volterra parameters dictated by the primary condition, the w-fields can then be expressed in terms of the u-fields exactly as do the primary w-currents which satisfy (Di-Francesco *et al.*, 1991)

$$w_s(z) = \psi^{-s}\tilde{w}_s(\tilde{z}). \tag{4.37}$$

To illustrate how things work, let us focus on solving Eq. (4.36) for the special case $n = 3$. We have

$$L_3(u) = \partial^3 + u_2\partial + u_3 \tag{4.38}$$

describing the Lax operator of the q-Boussinesq integrable system. Applying the Volterra gauge group symmetry Eqs. (4.36)–(4.38), by identifying

$$K(a)L_3(w) = L_3(u)K(a), \tag{4.39}$$

we find, after straightforward algebraic calculations, the following formulas for the first parameters a_1, a_2, a_3, a_4:

$$\begin{aligned}
\bar{q}^3 a_1 &= a_1 \\
a_2 + w_2 &= u_2 + \bar{q}^6 a_2 + \bar{q}^2(1 + \bar{q} + \bar{q}^2)a_1' \\
a_3 + w_3 + q^2 a_1 w_2 &= u_3 + \bar{q}^9 a_3 + \bar{q} a_1 u_2 + \bar{q}^4(1 + \bar{q} + \bar{q}^2)a_2' \\
&\quad + \bar{q}(1 + \bar{q} + \bar{q}^2)a_1'' \\
a_4 + q^3 a_1 w_3 - q^5 a_1 w_2' + q^4 a_2 w_3 &= a_1 u_3 + \bar{q}^2 a_2 u_2 + \bar{q}^{12} a_4 + \bar{q}^2(1 + \bar{q}
\end{aligned}$$

$$+\bar{q}^{2})a_2'' + \bar{q}^{6}(1 + \bar{q} + \bar{q}^{2})a_3' + a_1''a_1'u_2. \tag{4.40}$$

We also show that the remaining Volterra parameters a_j, $j \geq 2$, are constrained to satisfy

$$\begin{aligned} a_{j+3}(\bar{q}^{3(j+3)} - 1) &= a_1(-1)^{j-1}q^{3j+j\left(\frac{j-1}{2}\right)}w_3^{(j-1)} + a_1(-1)^j q^{2(j+1)+j\left(\frac{j+1}{2}\right)}w_2^{(j)} \\ &+ \sum_{i=0}^{\infty} a_{j-i}q^{3j+i\left(\frac{i+1}{2}\right)}\left(\sum_{k_1=0}^{i}\cdots\sum_{k_{j-i-1}=0}^{k_{j-i-2}} q^{\sum_{m=1}^{j-i-1}k_m}\right)w_3^{(i)} \\ &+ \sum_{i=0}^{\infty} a_{j-i+1}q^{2(j+1)+i\left(\frac{i+1}{2}\right)}\left(\sum_{k_1=0}^{i}\cdots\sum_{k_{j-i}=0}^{k_{j-i-1}} q^{\sum_{m=1}^{j-i-1}k_m}\right)w_2^{(i)} \\ &- \bar{q}^{j+1}(1 + \bar{q} + \bar{q}^{2})a_{j+1}'' - \bar{q}^{j+1}a_{j+1}u_2 - a_j''' \\ &- \bar{q}^{2(j+2)}(1 + \bar{q} + \bar{q}^{2})a_{j+2}' - a_j'u_2 - a_ju_3, . \end{aligned} \tag{4.41}$$

Consequently, we learn from Eq. (4.40) that the spin-1 Volterra gauge parameter a_1 vanishes naturally for arbitrary values of the parameter q. This leads to set

$$a_1 = 0 \tag{4.42a}$$

$$(1 - \bar{q}^{6})a_2 = u_2 - w_2 \tag{4.42b}$$

$$(1 - \bar{q}^{9})a_3 = u_3 - w_3 + \bar{q}^{4}(1 + \bar{q} + \bar{q}^{2})a_2' \tag{4.42c}$$

$$\begin{aligned} (\bar{q}^{9} - 1)a_4 &= q^4a_2w_3 - \bar{q}^{2}(1 + \bar{q} + \bar{q}^{2})a_2'' \\ &\quad - \bar{q}^{6}(1 + \bar{q} + \bar{q}^{2})a_3' - (a_1' + \bar{q}^{2}a_2)u_2, \end{aligned} \tag{4.42d}$$

with the constrains (4.41). Note that when $q = 1$, one recovers from Eqs. (4.42), a Volterra gauge orbit $K_{q=1}\{a_i\}$ in which the w_i-fields are seen as primary currents (Rachidi, xxxx).

Actually, our principal task is to make an appropriate choice of the Volterra parameters a_i such that w_i become primary conformal currents satisfying Eq. (4.37). Recall also that in the classical limit the analytic field u_2 behaves as a spin-2 field of 2D conformal field theory, coinciding with the w_2 current. Similarly, in the deformed case we can require w_2 to be proportional to u_2, which leads from Eq. (4.42b) to set

$$a_2 = \delta u_2, \tag{4.43}$$

where δ is an arbitrary constant for the moment. We then have

$$w_2 = u_2(1 - \delta(1 - \bar{q}^{6})). \tag{4.44}$$

Substituting Eq. (4.43) into Eq. (4.42c), we obtain

$$a_3 = \beta_1 u_3 + \beta_2 u_2', \tag{4.45}$$

where β_1 and β_2 are, for instance, arbitrary constants which can be fixed.

The resulting expression for the q-deformed w-current of spin 3 is

$$w_3 = u_3[1 + (\bar{q}^9 - 1)\beta_1] + u_2'\lfloor\bar{q}^4(1 + \bar{q} + \bar{q}^2)\delta + \beta_2(\bar{q}^9 - 1)\rfloor, \tag{4.46}$$

with the constraints equation (4.41) giving the remaining Volterra parameters a_j, ≥ 5.

$$\begin{aligned} a_4(\bar{q}^9 - 1) &= q^4 a_2 w_3 - \bar{q}^2(1 + \bar{q} + \bar{q}^2)a_2'' \\ &\quad - \bar{q}^6(1 + \bar{q} + \bar{q}^2)a_3' - (a_1' + \bar{q}^2 a_2)u_2 \\ a_{j+3}(\bar{q}^{3(j+3)} - 1) &= \sum_{i=0}^{3} a_{j-1} q^{3j+i\left(\frac{i+1}{2}\right)} \left(\sum_{k_1=0}^{i} \cdots \sum_{k_{j-i-1}=0}^{k_{j-i-2}} q^{\sum_{m=1}^{j-i-1} k_m}\right) w_3^{(i)} \\ &\quad + \sum_{i=0}^{3} a_{j-i+1} q^{2(j+1)+\left(\frac{i+1}{2}\right)} \left(\sum_{k_1=0}^{i} \cdots \sum_{k_{j-i}=0}^{k_{j-i-1}} q^{\sum_{m=1}^{j-1} k_m}\right) w_2^{(i)} \\ &\quad - \bar{q}^{j+1}(1 + \bar{q} + \bar{q}^2)a_{j+1}'' - \bar{q}^{j+1} a_{j+1} u_2 - a_j''' \\ &\quad - \bar{q}^{-2(j+2)}(1 + \bar{q} + \bar{q}^2)a_{j+2}' - a_j' u_2 - a_j u_3. \end{aligned} \tag{4.47}$$

Now, let us consider a conformal transformation of the spin-3 w-current [Eq. (4.46)]:

$$\tilde{w}_3 = \psi^3 w_3 + y_3, \tag{4.48}$$

where y_3 is a function of conformal spin 3 given by

$$\begin{aligned} y_3 &= \psi^2\psi'\{1 + 2\bar{q}^4(1 + \bar{q} + \bar{q}^2)\delta + (\bar{q}^9 - 1)\beta_1 + 2(\bar{q}^9 - 1)\beta_2\}u_2 \\ &\quad + \psi^3\left\{\left(S_{u^3}^{(3)} - \frac{\psi'}{\psi}S_{u^3}^{(3)}\right) - \left(2\frac{\psi'}{\psi}S_{u_2}^{(3)} + \partial S_{u_2}^{(3)}\right)\bar{q}^4(1 - \bar{q} + \bar{q}^2)\delta\right. \\ &\quad \left. + (\bar{q}^9 - 1)\left(S_{u_3}^{(3)} - \frac{\psi'}{\psi}S_{u_2}^{(3)}\right)\beta_1 - (\bar{q}^9 - 1)\left(2\frac{\psi'}{\psi}S_{u_2}^{(3)} + \partial S_{u_2}^{(3)}\right)\beta_2\right\}. \end{aligned}$$

Imposing the primary condition (4.37) implies the vanishing of y_3 from which one can derive a solution for the constants $\delta(q)$, $\beta_1(q)$, and $\beta_2(q)$ which are required to coincide in the classical limit with $\beta(1) = -1/6$, $\beta_1(1) = 0$, and $\beta_2(1) = 1/6$ respectively.

5. NOTE ON THE *su*(*n*)-TODA FIELD THEORY CONSTRUCTION

In we this section we set up some crucial ingredients toward building the q-deformed analogue of 2D su(n)-Toda like conformal field theory, using the previous analysis. The starting point consists in exploiting the correspondence which exists between the second Hamiltonian structure of integrable systems and the Virasoro conformal algebra which is the symmetry of 2D Liouville field theory.

Consider then the integrable q-KdV equation discussed previously in Section 3 and which we can conveniently take as follows (see Eq. (3.15)):

$$u_2 = \left(\frac{1+\bar{q}+\bar{q}^4}{1+\bar{q}^2}\right) u_2 u_2' - \frac{1+\bar{q}+\bar{q}^2}{(\bar{q}+1)} u_2'''. \tag{5.1}$$

Applying the Miura transformation (which connects the dynamical current u_2 with the scalar field $\varphi \equiv \varphi(z, \bar{z})$ to the q-deformed KdV Lax operator as follows:

$$L_2 = (\partial^2 + u_2) = (\partial + A)(\partial + B), \tag{5.2}$$

where A and B are spin-1 fields, which are constrained to satisfy

$$\begin{cases} A = -\bar{q}^B \\ AB + B' = u_2, \end{cases} \tag{5.3}$$

with $B' = (\partial B)$. A solution to this system is

$$\begin{cases} A = -\partial\varphi \\ B = q\partial\varphi, \end{cases} \tag{5.4}$$

which gives

$$u_2 = q(\partial^2\varphi - (\partial\varphi)^2). \tag{5.5}$$

This equation shows that u_2 is a q-deformed spin-2 current, which behaves like the stress-energy–momentum tensor of 2D Liouville conformal field theory. An important point is to look for the Lagrangian of this theory. Using the standard knowledge on conformal Liouville field theory (Alvarez-Gaumé and Gomez, 1991; Mansfield, 1982, 1983; Olive and Turok, 1986; Saidi and Sedra, 1993, 1994b,c; Sedra, 1998), we can set by analogy

$$S[\varphi] = \int d^2z \left\{\partial\varphi\bar{\partial}\varphi + \frac{2}{b}\ \exp(b\varphi)\right\}, \tag{5.6}$$

where the coefficient number b is shown to take the value $b = (1+\bar{q})$ (see Appendix D). We also show that the equation of motion which emerges from this action is nothing but the q-deformed 2D conformal Liouville equation given by

$$\partial\bar{\partial}\varphi - 2\bar{q}\, e^{(1+\bar{q})\varphi} = 0 \quad (\bar{q} = q^{-1}). \tag{5.7}$$

To obtain this equation, one must precise, as explicitly shown in Appendices C and D, that the Euler–Lagrange equations should be applied taking into account the previous analysis. The q-deformed form of the conserved current can be written as

$$T(\varphi) \equiv q\partial^2\varphi - q(\partial\varphi)^2, \tag{5.8}$$

whose conservation is assured by the equation of motion (5.7),

$$\bar{\partial}T(\varphi) = 0. \tag{5.9}$$

Note that this conservation law combined with Eq. (5.7) fixes the q-coefficient number $b = (1 + \bar{q})$ in the exponential Equation (5.6). Before closing this discussion some remarks are in order.

First note that the action (5.6) is conformally invariant and generalizes naturally the su(2) standard Liouville theory. As already known from the standard studies, the coefficient number in the exponential Liouville potential is closely connected with the Cartan matrix of some simple Lie algebra. An important task is to look for the interpretation of the coefficient $(\bar{q} + 1)$, appearing in our exponential, from the Lie algebraic point of view. Recall that this number coincides in the classical limit case with the number 2, which is nothing but the Cartan matrix of the su(2) Lie algebra.

However, the choice of our q-KdV Lax operator in Eq. (5.2) shows already the existence of an su(2) symmetry, which can also be recovered from the Liouville action. Indeed, if we redefine the scalar field φ to be

$$\Phi = \frac{\bar{q} + 1}{2}\varphi, \tag{5.10}$$

we can easily read the su(2) symmetry from the Liouville action. The latter becomes

$$S[\Phi] = \int d^2z \left\{\lambda\partial\Phi\bar{\partial}\phi + \frac{2}{\bar{q} + 1}\ \exp(2\phi)\right\} \tag{5.11}$$

upon introducing a parameter Λ, namely

$$\lambda = \left(\frac{\bar{q} + 1}{2}\right)^2. \tag{5.12}$$

The q-deformed Liouville equation of motion becomes

$$\partial\bar{\partial}\Phi - \bar{q}(\bar{q} + 1)\ \exp(2\Phi) = 0. \tag{5.13}$$

We can also think to generalize the above q-deformed su(2)-Liouville field theory to the su(n) conformal Toda field theory. We set for the moment

$$S_{\mathrm{su}}(n) - \mathrm{Toda} = \int \partial^2 z \left(\partial\phi\bar{\partial}\phi + \eta(q)\sum_{i=1}^{n-1}\exp(\alpha_i\phi)\right), \tag{5.14}$$

where $\phi = \sum_{j=1}^{n-1} \alpha_j \phi_j$ and α_j are the simple root of the su(n) Lie algebra whose Cartan matrix is defined as

$$K_{ij} = \alpha_i \alpha_j, \tag{5.15}$$

and where $\eta(q)$ is a function of the parameter q, which can easily be fixed, given the corresponding model in the generalized KdV hierarchy. More on this q-deformed Toda field theory construction may be a subject of future works.

6. CONCLUSION

We tried in this work to understand the behavior of 2D nonlinear integrable systems in the q-deformed case. For this reason, we started by generalizing some well-known results in the theory of formal pseudo-differential operators to the q-deformed case. The obtained results are applied to build the q-analogues of the generalized integrable q-KdV hierarchies whose first leading orders are the q-KdV and q-Boussinesq systems. We derived the dynamical equations of these deformed integrable hierarchies, leading in fact to the standard ones, once the q-parameter is fixed to be 1. We discussed how to transform in the deformed case the currents $u_j(z)$ under a conformal transformation. The results obtained showed a nontrivial behavior of these currents, which coincides naturally with the standard results upon setting $q = 1$. We discussed also the primary condition of these currents using the Volterra gauge group symmetry. In the last part of this work, devoted to the Toda field theory construction, we presented the q-analogue of the su(2) Liouville and su(n) Toda conformal field theories. Other algebraic properties are reported in the appendices.

APPENDIX A

Let $f(z)$ be an arbitrary analytic function of conformal spin $\Delta f = \tilde{f}$. Using Eq. (2.3) and the iterative action of the q-deformed derivative, we find

$$\partial f^n(z) = (1 + q^{-1\tilde{f}} + q^{-2\tilde{f}} + \cdots q^{-(n-1)\tilde{f}}) f' f^{n-1} + q^{-n\tilde{f}} f^n \partial, \tag{A1}$$

where n is a positive integer number. Setting $q = 1$ one recovers, once again, the ordinary derivation rule

$$\partial f^n(z) = n f' f^{n-1} + f^n \partial. \tag{A2}$$

A special choice of $f(z)$ in Eq. (A1) is given by $f(z) = z$ with $\tilde{z} = -1$,

$$\partial z^n = (1 + q + q^2 + \cdots + q^{n-1}) z^{n-1} + q^n z^n \partial, \tag{A3}$$

which reduces to Eq. (2.1) upon setting $n = 1$. For negative integer numbers we easily find

$$\partial f^{-n}(z) = -(q^{\tilde{f}} + q^{2\tilde{f}} + \cdots + q^{n\tilde{f}}) f' f^{n-1} + q^{n\tilde{f}} f^{-n} \partial, \tag{A4}$$

which becomes, upon setting $q = 1$,

$$\partial f^{-n}(z) = -n f' f^{-n-1} + f^{-n} \partial. \tag{A5}$$

As before, setting $f(z) = z$ we obtain

$$\partial z^{-n} = -(\overline{q} + \overline{q}^2 + \cdots + \overline{q}^n) z^{-n-1} + \overline{q}^n z^{-n} \partial. \tag{A6}$$

Furthermore, we note that for half integer powers of $f(z)$ we can obtain general formulas. The method to do this starts from setting

$$\partial f^{1/2} = \alpha(q) f' f^{-1/2} + \beta(q) f^{1/2} \partial, \tag{A7}$$

where $\alpha(q)$ and $\beta(q)$ are two arbitrary q-dependent functions that we can determine explicitly by the following trivial property:

$$\partial(f^{1/2} f^{1/2}) \equiv \partial(f). \tag{A8}$$

General formulas are given by

$$\partial f^{\frac{2n+1}{2}}(z) = \frac{\left(1 + \overline{q}^{\frac{\tilde{f}}{2}} + \overline{q}^{\frac{2\tilde{f}}{2}} + \cdots + \overline{q}^{\frac{-2n\tilde{f}}{2}}\right)}{(1 + \overline{q}^{\frac{\tilde{f}}{2}})} f' f^{\frac{2n+1}{2}} + \overline{q}^{\frac{(2n+1)\tilde{f}}{2}} f^{\frac{(2n+1)}{2}} \partial, \tag{A9}$$

and

$$\partial f^{\frac{-2n+1}{2}}(z) = \frac{-q^{\frac{\tilde{f}}{2}}\left(q^{\frac{\tilde{f}}{2}} + q^{\frac{2\tilde{f}}{2}} + q^{\frac{3\tilde{f}}{2}} \cdots + q^{\frac{(2n+1)\tilde{f}}{2}}\right)}{(1 + \overline{q}^{\frac{\tilde{f}}{2}})} f' f^{\frac{-(2n+3)}{2}} + q^{\frac{(2n+1)\tilde{f}}{2}} f^{\frac{-(2n+1)}{2}} \partial. \tag{A10}$$

These q-generalized results are important in discussing the q-deformed Lax evolution equations and the covariantization of q-differential Lax operators.

Before closing this appendix, note that the ring $R = \oplus_{k \in z} R_k$ defined in Eq. (2.2) is a commutative ring, which means that for each $u_k(z)$ and $u_i(z)$ belonging to R we have $u_k(z)u_i(z) = u_i(z)u_k(z)$. However, applying the q-Leibnitz rule (2.3), one can easily show the existence of a noncommutative structure in the space $\Xi_m^{(r,s)}$ of local and nonlocal q-differential operators. Indeed, let f and g be two arbitrary functions of conformal spin $\tilde{f}$ and $\tilde{g}$, with $fg = gf$,

$$(\partial f)g = f'g + \overline{q}^{\tilde{f}} fg' + \overline{q}^{(\tilde{f}+\tilde{g})} fg\partial, \tag{A11}$$

while

$$(\partial g)f = g'f + \overline{q}^{\tilde{g}} gf' + \overline{q}^{(\tilde{f}+\tilde{g})} gf\partial, \tag{A12}$$

which clearly shows that $(\partial f)g \neq (\partial g)f$ for $\tilde{f} \neq \tilde{g}$. Note that this noncommutativity property of f and g, with respect to the action of the q-derivative ∂_q, arises naturally from Eq. (2.3). Note also the important fact that when the function g is, for example, the nth power of the function f with $n \in R$, one can set $g = f^n$ which yields $\tilde{g} = n\tilde{f}$ and then

$$(\partial f)g = (\partial g)f, \tag{A13}$$

with $f' f^n = f^n f'$. One can then deduce that the two subspaces $R_{\tilde{f}}$ and $R_{\tilde{f}}$ of analytic functions $f(z)$ and $g(z)$ of conformal spin $\tilde{f}$ and $\tilde{g}$, respectively, do not commute under the action of the q-derivative ∂_q unless if there exists a relative integer $n \in Z$, such that $g = f^n$.

APPENDIX B

q-Deformed Boussinesq Equation

Using the same technique developed for the q-deformed KdV system, we present in this appendix a q-generalization of the Boussinesq integrable hierarchy (for a review, see Das, 1989).

Let

$$L_3 = \partial^3 + u_2\partial + u_3 \tag{B1}$$

be the Lax operator associated with the q-Boussinesq hierarchy, where u_2 and u_3 are two currents of conformal spin 2 and 3, respectively. Knowing that $(L_3^{1/3})^3 = L_3$ and the fact that $L_3^{1/3}$ is an object of conformal spin 1, we can set

$$L_3^{\frac{1}{3}} = \partial + au_2\partial^{-1} + (bu_3 - cu_2')\partial^{-2} + (du_2'' - eu_2^2 - fu_3')\partial^{-3} + \cdots, \tag{B2}$$

where the coefficients a, b, c, d, e, and f are given explicitly by

$$\begin{aligned}
a &= \frac{1}{1+\bar{q}^2+\bar{q}^4}\\
b &= \frac{1}{1+\bar{q}^3+\bar{q}^6}\\
c &= \frac{1+\bar{q}^2+\bar{q}^3}{(1+\bar{q}^2+\bar{q}^4)(1+\bar{q}^3+\bar{q}^6)}\\
d &= \frac{1}{(1+\bar{q}^2+\bar{q}^4)(1+\bar{q}^4+\bar{q}^8)}\left\{\frac{(1+\bar{q}^2+\bar{q}^3)(1+\bar{q}^3+\bar{q}^4)}{1+\bar{q}^3+\bar{q}^6} - 1\right\}\\
e &= \frac{1+q^2+\bar{q}^2}{(1+\bar{q}^2+\bar{q}^4)^2}
\end{aligned} \tag{B3}$$

$$f = \frac{1+\overline{q}^3+\overline{q}^4}{1+\overline{q}^3+\overline{q}^6}$$

so that

$$\left(L_3^{1/3}\right)_+ = \partial. \tag{B4}$$

Identifying the r.h.s. and l.h.s. of the following equation:

$$\frac{\partial L_3}{\partial t_1} = \left[\left(L_3^{\frac{1}{3}}\right)_+, L_3\right]_q, \tag{B5}$$

we obtain

$$\begin{aligned} u_2' &= u_2 \\ u_3' &= u_3, \end{aligned} \tag{B6}$$

which give the chiral wave equations for the Boussinesq hierarchy.
Similarly, the identification

$$\frac{\partial L_3}{\partial t}\left[\left(L_3^{\frac{2}{3}}\right)_+, L_3\right]_q \tag{B7}$$

with

$$\left(L_3^{\frac{2}{3}}\right)_+ = \partial^2 + a(\overline{q}^2+1)u_2 \tag{B8}$$

gives

$$\dot{u}_3 = u_3'' + a(1+\overline{q}^2)\left\{\alpha_{u_2}''' + \beta_{u_2u_2}\right\} \tag{B9a}$$

$$\dot{u} = u_2''\{1+\alpha_{a\overline{q}}^2(1+\overline{q}^2)(1+\overline{q}+\overline{q}^2)\} + \overline{q}^3(1+\overline{q})u_3' \tag{B9b}$$

$$\overline{q}^3 u_3'' = (1+\overline{q})\left\{1+\alpha_{a\overline{q}}^2(1+\overline{q}^2)\right\}u_2''', \tag{B9c}$$

where α and β are two arbitrary functions of the parameter q, which can be conveniently fixed in such a way that $\alpha = \beta = -1$ in the classical limit. Combining (B9a) and (B9c) we find

$$\dot{u}_3 = -q^2(1+q+a\alpha\overline{q}(1+\overline{q}^2))u_2''' + a\beta(1+\overline{q}^2)u_2u_2'. \tag{B10}$$

Moreover, note that (B9c) can be written as

$$\dot{u}_3 = \frac{-\overline{q}^3}{(1+\overline{q})(1+a\alpha\overline{q}^2(1+\overline{q}^2))}, \tag{B11}$$

which implies by virtue of (B9b) that

$$\dot{u}_2 = B(q,\alpha)u_3', \tag{B12}$$

with

$$B(q,\alpha) = \bar{q}^3(1+\bar{q}) - \frac{\bar{q}^3(1+a\alpha\bar{q}^2(1+\bar{q}^2)(1+\bar{q}+\bar{q}^2))}{(1+\bar{q})(1+a\alpha\bar{q}^2(1+\bar{q}^2))}. \tag{B13}$$

Equations (B10) and (B12) then give the q-deformed Boussines q equations. Setting $q = 1$ we recover the classical Boussineq equation, namely (Das, 1989)

$$\begin{aligned}\dot{u}_2 &= \frac{7}{2}u_3' \\ \dot{u}_3 &= -\frac{4}{3}u_3''' - \frac{2}{3}u_2u_2'.\end{aligned} \tag{B14}$$

Next we will show how the q-deformed Boussinesq equations (B10) and (B12) can be cast into a simple form. Indeed using straightforward algebraic computations, (B10) and (B12) simply become

$$\ddot{u}_2 = B(q,\alpha)\left\{x_t u_2'' + \frac{x_2}{1+\bar{q}^2}u_2^2\right\}, \tag{B15}$$

where

$$\begin{aligned}x_1 &= -\bar{q}^2(1+q+a\alpha q(1+\bar{q}^2)) \\ x_2 &= a\beta(1+\bar{q}^2).\end{aligned} \tag{B16}$$

For $q = 1$ we recover the standard Boussinesq equation, namely

$$\ddot{u}_2 = 2u_2'''' + \frac{1}{2}\left(u_2^2\right)'' \tag{B17}$$

APPENDIX C

q-Deformed Exponential

The exponential function $\exp(z)$ is also shown to take a q-deformed form. Indeed, from Eq. (2.7) we can extract the following prime derivative:

$$(z^n)' \equiv (\partial z^n) = \left(\sum_{i=0}^{n-1} q^i\right) z^{n-1}, \tag{C1}$$

and write the exponential $\exp(z)$ as follows:

$$\exp(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \tag{C2}$$

where we define the q-deformed factorial numbers as follows:

$$[n]_q! = 1.(q+1).(q^2+q+1)\cdots(q^{n-1}+q^{n-2}+\cdots+q+1). \tag{C3}$$

With this definition, we clearly see, from Eqs. (C1) and (C2) that

$$\partial \exp(z) \equiv \exp(z), \tag{C4}$$

with the observation that

$$\sum_{i=0}^{n-1} q^i = \frac{[n]_q!}{[(n-1)]_q!}. \tag{C5}$$

Note that one can generalize these definitions of the exponential for an arbitrary function $f(z)$ of conformal spin $\tilde{f}$ by exploiting the results established before.

APPENDIX D

Variational Principle and q-Deformed Euler–Lagrange Equations

Consider the q-deformed Liouville action which we can write as

$$S[\varphi] = \int d^2z \left\{ \partial\varphi\bar{\partial}\varphi + \frac{2}{b}\exp(b\varphi) \right\}, \tag{D1}$$

where a and b are q-dependent coefficients which can be determined using dimensional arguments and conservation of the induced conserved current. The variational principle applied to the q-Liouville action S reads as

$$\delta S[\varphi] = 0 \Leftrightarrow \int d^2z \left\{ \frac{\partial L}{\partial\varphi}\delta\varphi + \frac{\partial L}{\partial(\partial\varphi)}\delta(\partial\varphi) \right\} = 0, \tag{D2}$$

where the Largrangian is given by $L = \partial\varphi\bar{\partial}\varphi + \frac{2}{b}\exp(b\varphi)$ with $\partial = \partial_q$ and where the variation δ is required to satisfy $[\delta, \partial]_q = 0$, which means that $\partial\delta = \delta\partial$. Using these remarks and the fact that (by virtue of Eq. (2.3))

$$\partial\left(\frac{\partial L}{\partial(\partial\varphi)}\delta\varphi\right) \equiv \left(\frac{\partial L}{\partial(\partial\varphi)}\delta\varphi\right)' = \left(\frac{\partial L}{\partial(\partial\varphi)}\right)'\delta\varphi + \bar{q}^x\frac{\partial L}{\partial(\partial\varphi)}\partial(\delta\varphi), \tag{D3}$$

where x is the conformal dimension of $(\frac{\partial L}{\partial(\partial\varphi)})$, we obtain the following q-deformed Euler–Lagrange equation

$$\frac{\partial L}{\partial\varphi} - q^x\partial\frac{\partial L}{\partial(\partial\varphi)} = 0 \tag{D4}$$

for the q-Liouville Lagrangian density $L = \partial\varphi\bar{\partial}\varphi + \frac{2}{b}\exp(b\varphi)$. Performing simple algebraic computations, we find (see Appendix C)

$$\frac{\partial L}{\partial\varphi} = \frac{2}{b}\frac{\partial e^{b\varphi}}{\partial\varphi} = 2e^{b\varphi}$$

$$\partial\frac{\partial L}{\partial(\partial\varphi)} = \partial\bar{\partial}\varphi \tag{D5}$$

from which we easily derive the following q-deformed Liouville equation of motion:

$$2e^{b\varphi} - q^x \partial\bar{\partial}\varphi = 0. \tag{D6}$$

On the other hand, using dimensional arguments, we remark that $x = 1$ as the conformal dimension of the Lagrangian is $\tilde{L} = 2$. To determine the coefficient constant b we use the conservation of the q-deformed current (5.8), namely

$$T(\varphi) = q\partial^2\varphi - q(\partial\varphi)^2. \tag{D7}$$

We have

$$0 = \bar{\partial}T(\varphi) = q\partial(\partial\bar{\partial}\varphi) - q\bar{\partial}(\partial\varphi)^2, \tag{D8}$$

which fixes the value of the coefficient b, namely $b = (1 + \bar{q})$ with $\tilde{\varphi} = 0$ and

$$\bar{\partial}(\partial\varphi)^2 = (1 + \bar{q})\partial\varphi\partial\bar{\partial}\varphi, \tag{D9}$$

as shown in previous computations. Finally, we have

$$\partial\bar{\partial}\varphi - 2\bar{q}e^{(1+\bar{q})\varphi} = 0 \quad (\bar{q} = q^{-1}). \tag{D10}$$

Setting $q = 1$ one recovers the well-known Liouville equation $\partial\bar{\partial}\varphi = 2e^{2\varphi}$ associated to the Liouville Lagrangian $L = \partial\varphi\bar{\partial}\varphi + \exp(2\varphi)$.

APPENDIX E

q-Deformed Commutator and Compatibility Condition

The use of the q-deformed commutator (3.8) instead of the usual one, namely $[L, B] = LB - BL$ which is nothing but the $q = 1$ limit of Eq. (3.8), implies a nontrivial consideration of the Lax evolution equation (3.2) in terms of the two compatibility equations. To be more precise let us recall how these equations give rise to the standard evolution Lax equation (for $q = 1$) for arbitrary Lax pair L, B. The compatibility equations are given by the following system of linear equations:

$$\begin{aligned} L\Psi &= \lambda\Psi \\ B\Psi &= \frac{\partial\Psi}{\partial t}. \end{aligned} \tag{E1}$$

We have

$$BL\Psi = B\lambda\Psi = \lambda B\Psi = \lambda\frac{\partial\Psi}{\partial t} = \frac{\partial\lambda\Psi}{\partial t} = \frac{\partial L\Psi}{\partial t}, \tag{E2}$$

which also gives

$$BL\Psi = \frac{\partial L\Psi}{\partial t} = \frac{\partial L}{\partial t}\Psi + L\frac{\partial\Psi}{\partial t} = \frac{\partial L}{\partial t}\Psi + LB\Psi. \tag{E3}$$

We then have

$$[B, L]\Psi = (BL - LB)\Psi = \frac{\partial L}{\partial t}\Psi \Leftrightarrow [B, L] = \frac{\partial L}{\partial t}. \tag{E4}$$

In the q-deformed case, the situation is not trivial, since the commutator is indispensable to ensure this compatibility is q-deformed. In fact, let us consider for simplicity the q-differential Lax pairs L_2 and $H_{2n+1} = (L_2^{2n+1/2})_+$ required to satisfy by analogy the Lax evolution equation (3.2)

$$\frac{\partial L_2}{\partial t_{2n+1}} = [H_{2n+1}, L_2]_q, \tag{E5}$$

where the q-deformed commutator is defined in Eq. (3.8). As suspected, by simply performing algebraic computations, we obtain

$$[H_1, L_2]_q = H_1 L_2 - \bar{q}^2 L_2 H_1 + (\bar{q}^2 - 1)\partial^3 + \cdots \tag{E6a}$$

$$[H_3, L_2]_q = H_3 L_2 - \bar{q}^6 L_2 H_3 + (\bar{q}^6 - 1)\partial^5 + \cdots \tag{E6b}$$

$$[H_5, L_2]_q = H_5 L_2 - \bar{q}^{10} L_2 H_5 + (\bar{q}^{10} - 1)\partial^7 + \cdots \tag{E6c}$$

$$[H_7, L_2]_q = H_7 L_2 - \bar{q}^{14} L_2 H_7 + (\bar{q}^{14} - 1)\partial^9 + \cdots \tag{E6d}$$

$$\vdots$$

results, which can be generalized for arbitrary order n of the q-KdV hierarchy as follows:

$$[H_{2n+1}, L_2]_q = H_{2n+1} L_2 - \bar{q}^{2(2n+1)} L_2 H_{2n+1} + (\bar{q}^{2(2n+1)} - 1)\partial^{2n+3} + \cdots. \tag{E6e}$$

The terms $(\bar{q}^{2(2n+1)} - 1)\partial^{2n+3} + \cdots)$ in (E.6) are extra nonlinear q-differential operators proportional to $(\bar{q} - 1)$. These extra terms vanish in the standard limit $\bar{q} = 1$ to give rise to the standard commutator (E.4)

$$[H_{2n+1}, L_2]_{q=1} = H_{2n+1} L_2 - L_2 H_{2n+1}. \tag{E7}$$

The important remark at this level is that if the compatibility equations exist they must be highly nonlinear with a dependence in $\bar{q}$ as they should take into account the presence of the nonlinear extra terms in the q-deformed commutators (E.6). The possibility to write the two compatibility linear equations can emerge naturally as a $\bar{q} = 1$ limit of the previous equations.

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